Asymptotic quasinormal mode spectrum of rotating black holes

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(February 7, 2008)

Motivated by novel results in the theory of black-hole quantization, we study *analytically* the quasinormal modes (QNM) of (*rotating*) Kerr black holes. The black-hole oscillation frequencies tend to the asymptotic value $\omega_n = m\Omega + i2\pi T_{BH} n$ in the $n \to \infty$ limit. This simple formula is in agreement with Bohr's correspondence principle. Possible implications of this result to the area spectrum of quantum black holes are discussed.

Gravitational waves emitted by a perturbed black hole are dominated by 'quasinormal ringing', damped oscillations with a *discrete* spectrum (see e.g., [1] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [2–5].

Being the characteristic 'sound' of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters (especially, the mass and angular momentum of the central black hole). This has motivated a flurry of activity with the aim of computing the spectrum of oscillations.

The ringing frequencies are located in the complex frequency plane characterized by Im$\omega > 0$. It turns out that for a given angular harmonic index l there exist an infinite number of quasinormal modes, for $n = 0, 1, 2, \ldots$, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [6,7]. On the other hand, the real part of the frequencies approaches an asymptotic *constant* value.

The QNM have been the subject of much recent attention (see e.g., [8–33] and references therein), with the hope that these classical frequencies may shed some light on the elusive theory of quantum gravity. These recent studies are motivated by an earlier work of Hod [34].

Few years ago I proposed to apply *Bohr's correspondence principle* in order to determine the value of the fundamental area unit in a quantum theory of gravity. It is useful to recall that in the early development of quantum mechanics, Bohr suggested a correspondence between classical and quantum properties of the Hydrogen atom, namely that "transition frequencies at large quantum numbers should equal classical oscillation frequencies". The black hole is in many senses the "Hydrogen atom" of General relativity. I therefore suggested [34] a similar usage of the discrete set of black-hole frequencies in order to shed some light on the *quantum* properties of a black hole. There is, however one important difference between the Hydrogen atom and a black hole: while a (classical) atom emits radiation spontaneously according to the (classical) laws of electrodynamics, a *classical* black hole does not emit radiation. This crucial difference hints that one should look for the highly damped black-hole free oscillations [let $\omega =$Re$\omega + i$Imω, then $\tau \equiv (\text{Im}\omega)^{-1}$ is the effective relaxation time for the black hole to return to a quiescent state after emitting gravitational radiation. Hence, the relaxation time $\tau \to 0$ as Im$\omega \to \infty$, implying no radiation emission, as should be the case for a classical black hole].

Leaver [6] was the first to address to problem of computing the black hole highly damped ringing frequencies. Nollert [35] found numerically (see also [36]) that the asymptotic behavior of the ringing frequencies of a Schwarzschild black hole is given by (we normalize $G = c = 2M = 1$)

$$\omega_n = 0.0874247 + \frac{i}{2}\left(n + \frac{1}{2}\right) \ , \tag{1}$$

In [34] it was realized that this asymptotic value equals $\ln 3/(4\pi)$. A heuristic picture (based on thermodynamic and statistical physics arguments) was suggested trying to explain this fact [34]. Most recently, Motl [10] has given an analytical proof for this equality.

Using the relation $A = 16\pi M^2$ for the surface area of a Schwarzschild black hole, and $\Delta M = E = \hbar\omega$ one finds $\Delta A = 4\ell_P^2 \ln 3$ with the emission of a quantum, where ℓ_P is the Planck length. Thus, we concluded that the area spectrum of the quantum Schwarzschild black hole is given by

$$A_n = 4\ell_P^2 \ln 3 \cdot n \quad ; \quad n = 1, 2, \ldots \ . \tag{2}$$

This result is remarkable from a statistical physics point of view. In the spirit of Boltzmann-Einstein formula in statistical physics, Mukhanov and Bekenstein [37–40] relate $g_n \equiv exp[S_{BH}(n)]$ to the number of microstates of the black hole that correspond to a particular external macrostate. In other words, g_n is the degeneracy of the nth area eigenvalue. The accepted thermodynamic relation between black-hole surface area and entropy $S_{BH} = \frac{1}{4}A$ [41], combined with the requirement that g_n has to be an integer for every n, actually enforce a factor of the form $4\ln k$ (with $k = 2, 3, \ldots$) in Eq. (2). It turns out that the value $k = 3$ is the only one compatible both with the area-entropy thermodynamic relation for black hole, and with Bohr's correspondence principle as well.

It should be emphasized that the asymptotic behavior of the black hole ringing frequencies is known only for the simplest case of a Schwarzschild black hole. Less is known about the corresponding QNM spectrum of the (rotating) Kerr black hole, in which case numerical calculations are limited to $n \leq 50$ [6,42,43,17,32]. This is a direct consequence of the numerical complexity of the problem. The aim of the present Letter is to study analytically the asymptotic quasinormal mode spectrum of generic Kerr black holes.

The black-hole perturbations are governed by the well-known Regge-Wheeler equation [44] in the case of a Schwarzschild black hole, and by the Teukolsky equation [45] for the (rotating) Kerr black hole. The black hole QNM correspond to solutions of the wave equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [46]. Such boundary conditions single out *discrete* solutions ω (assuming a time dependence of the form $e^{i\omega t}$). The solution to the radial Teukolsky equation may be expressed as [6]

$$R_{lm} = e^{i\omega r}(r-r_-)^{-1-s+i\omega+i\sigma_+}(r-r_+)^{-s-i\sigma_+} \Sigma_{n=0}^{\infty} d_n \Big(\frac{r-r_+}{r-r_-}\Big)^n \ , \tag{3}$$

where $r_\pm = M \pm (M^2-a^2)^{1/2}$ are the black hole (event and inner) horizons ($a = J/M$ is the black hole angular momentum per unit mass). The field spin-weight parameter s takes the values $0, -1, -2$ respectively, for scalar, electromagnetic and gravitational fields.

The sequence of expansion coefficients $\{d_n : n = 1, 2, \ldots\}$ is determined by a recurrence relation:

$$\alpha_n d_{n+1} + \beta_n d_n + \gamma_n d_{n-1} = 0 \ , \tag{4}$$

where $d_0 = 1$ and

$$\alpha_0 d_1 + \beta_0 d_0 = 0 \ . \tag{5}$$

The recursion coefficients are given by [6]

$$\alpha_n = n^2 + (c_0+1)n + c_0 \ , \tag{6}$$

$$\beta_n = -2n^2 + (c_1+2)n + c_3 \ , \tag{7}$$

and

$$\gamma_n = n^2 + (c_2-3)n + c_4 - c_2 + 2 \ , \tag{8}$$

where the intermediate constants c_n are defined by

$$c_0 = 1 - s - i\omega - \frac{2i}{b}\Big(\frac{\omega}{2} - am\Big) \ , \tag{9}$$

$$c_1 = -4 + 2i\omega(2+b) + \frac{4i}{b}\Big(\frac{\omega}{2} - am\Big) \ , \tag{10}$$

$$c_2 = s + 3 - 3i\omega - \frac{2i}{b}\Big(\frac{\omega}{2} - am\Big) \ , \tag{11}$$

$$c_3 = \omega^2(4+2b-a^2) - 2am\omega - s - 1 + (2+b)i\omega - A_{lm} + \frac{4\omega+2i}{b}\Big(\frac{\omega}{2} - am\Big) \ , \tag{12}$$

and

$$c_4 = s + 1 - 2\omega^2 - (2s+3)i\omega - \frac{4\omega+2i}{b}\Big(\frac{\omega}{2} - am\Big) \ , \tag{13}$$

where $b = (1-4a^2)^{\frac{1}{2}}$ and the separation constants A_{lm} are given by an independent recurrence relation [6].

The quasinormal frequencies are determined by the requirement that the series in Eq. (3) absolutely convergent, i.e. that is Σd_n exists and is finite [6].

The *physical* content of the expansion coefficients becomes clear when they are expressed in terms of the black-hole physical parameters T_{BH} and Ω, in which case they obtain a surprisingly simple (and compact) form [47]

$$\alpha_n = -i\frac{n+1}{2\pi T_{BH}}\Big[\omega - m\Omega + i2\pi T_{BH}(n+1-s)\Big] \ , \tag{14}$$

and

$$\gamma_n = -\frac{2\omega+in}{2\pi T_{BH}}\Big[\omega - m\Omega + i2\pi T_{BH}(n+s)\Big] \ . \tag{15}$$

Here $T_{BH} = (r_+ - r_-)/A$ is the Bekenstein-Hawking temperature, and $\Omega = 4\pi a/A$ is the angular velocity of the black-hole horizon. The explicit expression of the $\{\beta_n\}$ coefficients is not important for the analysis, but for later purposes it is important to note that $\beta_n \to -(a\omega)^2$ in the $n \to \infty$ limit with $\omega \sim -i2\pi T_{BH} n$ [48].

Quasinormal frequencies of the Kerr black hole. Taking cognizance of Eqs. (14) and (15), one finds that in the $n \to \infty$ limit with $\omega \sim -i2\pi T_{BH} n$, the α_n and γ_n coefficients are of order $O(n)$, while $\beta_n = O(n^2)$. The term $\alpha_n d_{n+1}$ in Eq. (4) is therefore negligible as compared to the other two terms $\beta_n d_n$ and $\gamma_n d_{n-1}$ [one finds $\alpha_n d_{n+1}/\beta_n d_n = O(a^{-4}b^{-4}n^{-2})$. The asymptotic limit therefore requires $n \gg (ab)^{-2}$]. If $\gamma_N = 0$ for some integer N, then *all* d_n with $n \geq N$ values would vanish, implying the convergence of the series $\sum d_n$. Taking cognizance of Eq. (15), one finds that $\gamma_N = 0$ for $\omega_N = m\Omega - i2\pi T_{BH} N$. Thus, the asymptotic quasinormal frequencies of a rotating Kerr black hole are given by the simple relation

$$\omega_n = m\Omega - i2\pi T_{BH} n \ . \tag{16}$$

We note that this asymptotic formula is in agreement with recent numerical results [32] for the $l = m = 2$ gravitational perturbations. However, the authors of [32]

cannot exclude the possibility that their numerical calculations for other perturbations actually break down *before* reaching the asymptotic regime [32].

Taking cognizance of the first law of black-hole thermodynamics

$$\Delta M = T_{BH}\Delta S + \Omega \Delta J \ , \qquad (17)$$

one finds that the asymptotic frequency corresponds to $\Delta S = \Delta A = 0$ [32]. In other words, the application of Bohr's correspondence principle to black-hole physics is *consistent* with the first law of black-hole thermodynamics.

Quasinormal frequencies of a Schwarzschild black hole. An analysis along the same lines as before provides a simple and elegant way to obtain the asymptotic quasinormal spectrum of electromagnetic perturbations of the spherically symmetric Schwarzschild black hole. One finds that in this case $\alpha_n = O(n)$ in the $n \to \infty$ limit with $\omega \sim -in/2$, while the β_n and γ_n coefficients are of order $O(n)$. Thus, the term $\beta_n d_n$ in Eq. (4) is negligible as compared to the other two terms $\alpha_n d_{n+1}$ and $\gamma_n d_{n-1}$ [one finds $\beta_n d_n/\alpha_n d_{n+1} = O(n^{-\frac{1}{2}})$]. If $\gamma_N = 0$ for some even (odd) integer N, then *all* d_n with even (odd) $n > N$ values would vanish. Taking cognizance of Eq. (15), one finds that for $\omega_N = -iN/2$ and $s = -1$ (electromagnetic perturbations), *both* γ_N and γ_{N+1} vanish, implying $d_N = 0$ for all $n > N$ values. This would guarantee the convergence of the series $\sum d_n$ (the same conclusion holds true for all odd integer values of the parameter $|s|$). The electromagnetic ($s = -1$) quasinormal frequencies of a Schwarzschild black hole are therefore given asymptotically by the simple relation

$$\omega_n = -in/2 \ . \qquad (18)$$

which agrees with the analysis of [10].

In summary, we have studied analytically the quasinormal mode spectrum of rotating Kerr black holes. The oscillation frequencies tend to the asymptotic value $\omega = m\Omega + i2\pi T_{BH} n$ in the $n \to \infty$ limit. This formula is consistent with Bohr's correspondence principle, and gives further support to its applicability in the quantum theory of gravitation.

ACKNOWLEDGMENTS

I thank a support by the Dr. Robert G. Picard fund in physics. This research was supported by grant 159/99-3 from the Israel Science Foundation.

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